An Extrapolation of Operator Valued Dyadic Paraproducts

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Abstract We consider the dyadic paraproducts π_φ on $\mathbb{T}$ associated with an $\mathcal{M}$-valued function φ. Here $\mathbb{T}$ is the unit circle and $\mathcal{M}$ is a tracial von Neumann algebra. We prove that their boundedness on $L^p(\mathbb{T}, L^p(\mathcal{M}))$ for some $1 < p < \infty$ implies their boundedness on $L^p(\mathbb{T}, L^p(\mathcal{M}))$ for all $1 < p < \infty$ provided φ is in an operator-valued BMO space. We also consider a modified version of dyadic paraproducts and their boundedness on $L^p(\mathbb{T}, L^p(\mathcal{M}))$.

1 Introduction

Let $(\mathbb{T}, \sigma_k, dt)$ be the unit circle with Haar measure and the usual dyadic filtration. Consider a function φ defined on $\mathbb{T}$. The dyadic paraproduct associated with φ, denoted by π_φ, is the operator on $L^2(\mathbb{T})$ defined as

$$\pi_\varphi(f) = \sum_k (d_k\varphi)(E_{k-1}f), \quad \forall f \in L^2(\mathbb{T}). \tag{1.1}$$

Here $E_k f$ is the conditional expectation of f with respect to σ_k, i.e. the unique σ_k-measurable function such that

$$\int_A E_k f dt = \int_A f dt, \quad \forall A \in \sigma_k.$$

And $d_k\varphi$ is defined to be $E_k\varphi - E_{k-1}\varphi$. It is not hard to check that the adjoint operator of π_φ is given as

$$(\pi_\varphi)^*(f) = \sum_k (d_k\bar{\varphi})(d_k f), \quad \forall f \in L^2(\mathbb{T}),$$

where $\bar{\varphi}$ is the complex conjugate of φ. We can of course consider the extension of π_φ on $L^p(\mathbb{T})$ for all $1 < p < \infty$.

A modified version of paraproducts Λ_φ is defined as

$$\Lambda_\varphi(f) = \sum_k (d_k\varphi)(E_k f).$$

Λ_φ is also called the Haar multiplier. It is easy to see that

$$\Lambda_\varphi = \pi_\varphi + (\pi_{\bar{\varphi}})^*.$$

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Paraproducts are usually considered as dyadic singular integrals and play important roles in the classical analysis. Like the singular integrals, dyadic paraproducts have the extrapolation property that their boundedness on L^p for some $1 < p < \infty$ implies their boundedness on L^p for all $1 < p < \infty$. In fact, π_φ's operator bound on L^p are equivalent to the dyadic BMO norm of φ's for all $1 < p < \infty$. The extrapolation property of paraproducts plays essential roles in the proof of many classical theorems, such us $T(1)$ theorem.

We'd like to consider the generalization of this extrapolation property of paraproducts in the noncommutative setting. Let $\mathcal{M}$ be a von Neumann algebra equipped with a semifinite normal faithful trace τ, and let $L^p(\mathcal{M})$ be the associated noncommutative L^p-space, $1 \leq p \leq \infty$ (see the next section for the definition). In particular, if $\mathcal{M} = B(\ell^2)$ equipped with the usual trace Tr, we get the Schatten p-class S^p. Let $L^p(\mathbb{T}; L^p(\mathcal{M}))$ denotes the usual L^p-space of Bochner p-integrable functions on the unit circle $\mathbb{T}$ with values in $L^p(\mathcal{M})$. We consider paraproducts π_φ (resp. Λ_φ) associated with a $\mathcal{M}$-valued function φ defined as same as in (1.1) (resp. (1.2)) but for $f \in L^p(\mathbb{T}; L^p(\mathcal{M}))$. We look for the property that π_φs' boundedness on $L^2(\mathbb{T}; L^2(\mathcal{M}))$ implies their boundedness on $L^p(\mathbb{T}; L^p(\mathcal{M}))$ for all $1 < p < \infty$. This is influenced and benefited by the rapid development of the study of noncommutative martingales and operator valued harmonic analysis during the last decay (see [11], [16], [17], [1], [13] and [14]). There, L^2 bounds of operator-valued paraproducts have been deeply studied. In [13], a partial result of the desired "extrapolation" property is proved by the author by considering π_φ and π_{φ^*} jointly. But, contrary to the classical case, we know that the operator-valued martingale transform fails the "extrapolation" property.

The missing of a Calderón-Zygmund decomposition argument imposes one of the main difficulties to prove such "extrapolation" properties in the noncommutative setting. Very recently, J. Parcet (see [18]) studied an analogue of Calderón-Zygmund decomposition for operator-valued functions. But its application to weak (1.1) inequality limits to singular integral operators with operator-valued "commuting" kernels. We should also point out the difference between our point of view for "extrapolation" and that of study on singular integral operators on Banach space valued L^p spaces, where "extrapolation" means that the boundedness of singular integral operators on $L^2(X)$ implies their boundedness on $L^p(X)$ for all $1 < p < \infty$ for a fixed Banach space X. Hytönen and Weis (see [5], [6]) recently proved this for singular integrals with B(X)-valued kernels satisfying certain R-Boundedness estimate. One can easily see the different meaning of 2 "extrapolations" in the particular case that $\mathcal{M} = B(\ell^2)$, $X = S_2$. In this particular case, we look for condition that the boundedness of the singular integral operators on $L^2(S_2)$ implies their boundedness on $L^p(S_p)$ for all $1 < p < \infty$ while the study on Banach space valued singular integrals considers the condition that the boundedness on $L^2(S_2)$ implies $L^p(S_2)$ for all $1 < p < \infty$.

Our main results are the following:

Theorem 1.1 *We have*

$$||\Lambda_\varphi||_{L^p(\mathbb{T},L^p(\mathcal{M}))\to L^p(\mathbb{T},L^p(\mathcal{M}))} \leq c_p||b||_{\mathrm{BMO}_\mathcal{M}}.$$

The $p = 2$ case of Theorem 1.1 is due to O. Blasco and S. Pott (see [1]).

Theorem 1.2 *For $\varphi \in \mathrm{BMO}_{\mathcal{M}}(\mathbb{T}, \mathcal{M})$, assume π_φ is bounded on $L^p(\mathbb{T}, L^p(\mathcal{M}))$ for some $1 < p < \infty$, then it is bounded on $L^p(\mathbb{T}, L^p(\mathcal{M}))$ for all $1 < p < \infty$.*

Note in the classical case (when $\mathcal{M} = \mathbb{C}$), the assumption $\varphi \in \mathrm{BMO}_{\mathcal{M}}(\mathbb{T}, \mathcal{M})$ correspondences to the standard "Caledéron Zygmund" condition for the kernels of singular integrals and is implied by the boundedness of π_φ on L^p for any p. Thus it is not necessary to assume it in the classical case.

2 Preliminaries

2.1 Noncommutative L^p-spaces.

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $\mathcal{S}_+$ be the set of all positive $x \in \mathcal{M}$ such that $\tau(\mathrm{supp}(x)) < \infty$, where $\mathrm{supp}(x)$ denotes the support of x, i.e. the least projection $e \in \mathcal{M}$ such that $ex = x$. Let $\mathcal{S}$ be the linear span of $\mathcal{S}_+$. Note that $\mathcal{S}$ is an involutive strongly dense ideal of $\mathcal{M}$. For $0 < p < \infty$ define

$$\|x\|_p = \left(\tau(|x|^p)\right)^{1/p}, \quad x \in \mathcal{S},$$

where $|x| = (x^*x)^{1/2}$, the modulus of x. One can check that $\|\cdot\|_p$ is a norm or p-norm on $\mathcal{S}$ according to $p \geq 1$ or $p < 1$. The corresponding completion is the noncommutative L^p-space associated with $(\mathcal{M},\ \tau)$ and is denoted by $L^p(\mathcal{M})$. By convention, we set $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm. The elements of $L^p(\mathcal{M})$ can be also described as measurable operators with respect to $(\mathcal{M},\ \tau)$.

We refer to [24] for more information and for more historical references on noncommutative L^p-spaces. In the sequel, unless explicitly stated otherwise, $\mathcal{M}$ will denote a semifinite von Neumann algebra and τ a normal semifinite faithful trace on $\mathcal{M}$.

We have the following Hölder's inequality and duality result,

$$\begin{aligned} ||fg||_{L^r(\mathcal{M})} &\leq ||f||_{L^p(\mathcal{M})}||g||_{L^q(\mathcal{M})}, \quad && \frac{1}{p} + \frac{1}{q} = \frac{1}{r},\ 0 < p, q, r \leq \infty, && (2.2) \\ (L^p(\mathcal{M}))^* &= L^q(\mathcal{M}), && \frac{1}{p} + \frac{1}{q} = 1,\ 1 \leq p < \infty. && (2.3) \end{aligned}$$

Let H be a Hilbert space and $B(H)$ the space of bounded operators on H. If $\mathcal{M} = B(H)$ equipped with the usual trace Tr, then the associated L^p-spaces are the usual Schatten classes $S^p(H)$ based on H. If $H = \ell^2$, $S^p(H)$ is denoted by S^p. It is convenient to represent the elements of S^p by infinite matrices.

On the other hand, let $\mathcal{M}$ be commutative, say, $\mathcal{M} = L^\infty(\Omega, \mu)$ for a measure space (Ω, μ). With τ equal to the integral against μ, we then recover the usual L^p-spaces $L^p(\Omega)$. This example can be extended to the setting of operator-valued functions. Let

$(\mathcal{N},\ \nu)$ be another von Neumann algebra with a normal semifinite faithful trace ν. Let $\mathcal{M} = L^\infty(\Omega)\bar{\otimes}\mathcal{N}$ be the tensor product von Neumann algebra, equipped with the tensor product trace. Then for every $p < \infty$ the space $L^p(\mathcal{M})$ coincides with the usual L^p-space $L^p(\Omega; L^p(\mathcal{N}))$ of Bochner p-integrable functions on Ω with values in $L^p(\mathcal{N})$. We will use this example in the particular case where $\Omega = \mathbb{T}$ is equipped with Haar measure.

We also need the following inequalities. The proof of them is quite simple although one of them looks "wrong" at first glance.

Lemma 2.3 *For $(a_k)_{k=1}^m \in L^p(\mathcal{M})), (b_k)_{k=1}^m \in L^q(\mathcal{M})$, We have*

$$||\sum_{k=1}^m a_k^* b_k||_{L^1(\mathcal{M})} \leq ||(\sum_k |a_k|^2)^{\frac{1}{2}}||_{L^p(\mathcal{M})}||(\sum_k |b_k|^2)^{\frac{1}{2}}||_{L^q(\mathcal{M})} \tag{2.4}$$

and

$$||\sum_{k=1}^m a_k^* b_k||_{L^1(\mathcal{M})} \leq ||(\sum_k |a_k^*|^2)^{\frac{1}{2}}||_{L^p(\mathcal{M})}||(\sum_k |b_k^*|^2)^{\frac{1}{2}}||_{L^q(\mathcal{M})} \tag{2.5}$$

for all $1/p + 1/q = 1$.

Proof. (2.4) is easily followed by Hölder's inequality. We embed $(a_k)_{k=1}^m$ (resp. $(b_k)_{k=1}^m$) into the first row (resp. column) of $M_m \otimes \mathcal{M}$ (the matrices with $\mathcal{M}$ valued coifficents) and get

$$\begin{aligned} ||\sum_{k=1}^m a_k^* b_k||_{L^1(\mathcal{M})} &= ||(\sum_{k=1}^m a_k^* \otimes e_{1,k})(\sum_{k=1}^m b_k \otimes e_{k,1})||_{L^1(M_m\otimes\mathcal{M})} \\ &\leq ||\sum_{k=1}^m a_k^* \otimes e_{1,k}||_{L^p(\mathcal{M})}||\sum_{k=1}^m b_k \otimes e_{k,1}||_{L^q(M_m\otimes\mathcal{M})} \\ &= ||(\sum_{k=1}^m |a_k|^2)^{\frac{1}{2}}||_{L^p(\mathcal{M})}||(\sum_{k=1}^m |b_k|^2)^{\frac{1}{2}}||_{L^q(\mathcal{M})} \end{aligned}$$

For (2.5), we have

$$\begin{aligned} ||\sum_{k=1}^m a_k^* b_k||_{L^1(\mathcal{M})} &= \sup_{v,||v||_{\mathcal{M}}\leq 1} |\tau \sum_{k=1}^m v a_k^* b_k| \\ &= \sup_{v,||v||_{\mathcal{M}}\leq 1} |\tau \sum_{k=1}^m b_k (v a_k^*)| \\ &= \sup_{v,||v||_{\mathcal{M}}\leq 1} |\tau \sum_{k=1}^m (b_k^*)^* (v a_k^*)| \\ &\leq \sup_{v,||v||_{\mathcal{M}}\leq 1} ||\sum_{k=1}^m (b_k^*)^* (v a_k^*)||_{L^1(\mathcal{M})}. \end{aligned}$$

Now use (2.4), we get

$$\begin{aligned}
\|\sum_{k=1}^{m} a_k^* b_k\|_{L^1(\mathcal{M})} &\leq \sup_{v, \|v\|_{\mathcal{M}} \leq 1} \|(\sum_k |b_k^*|^2)^{\frac{1}{2}}\|_{L^q(\mathcal{M})} \|(\sum_k |v a_k^*|^2)^{\frac{1}{2}}\|_{L^p(\mathcal{M})} \\
&\leq \|(\sum_k |a_k^*|^2)^{\frac{1}{2}}\|_{L^p(\mathcal{M})} \|(\sum_k |b_k^*|^2)^{\frac{1}{2}}\|_{L^q(\mathcal{M})}. \blacksquare
\end{aligned}$$

2.2 Operator valued BMO spaces

We need 2 kinds of operator-valued dyadic BMO spaces: $\mathcal{BMO}_{cr}(\mathbb{T}, \mathcal{M})$ and $\mathrm{BMO}_{\mathcal{M}}(\mathbb{T}, \mathcal{M})$.

The space $\mathcal{BMO}_{cr}(\mathbb{T}, \mathcal{M})$

The operator-valued BMO spaces $\mathcal{BMO}_{cr}(\mathbb{T}, \mathcal{M})$ have been studied in [16], [23] , [15], [8] and [13] in various context. We recall its definition in our setting. For an $\mathcal{M}$-valued function φ defined on $\mathbb{T}$, define

$$\|\varphi\|_{\mathrm{BMO_c}} = \sup_m \Big\{ \|E_m \sum_{k=m+1}^{\infty} (d_k \varphi)^* (d_k \varphi)\|_{\mathcal{M}}^{\frac{1}{2}} \Big\},$$

where, again, E_m is the conditional expectation with respect to the usual dyadic filtration and $d_k\varphi$ is the martingale difference $E_k\varphi - E_{k-1}\varphi$. It is not hard to check that

$$\begin{aligned}
\|\varphi\|_{\mathrm{BMO_c}} &= \sup_I \| \int_I |\varphi - \varphi_I|^2 dt \|_{\mathcal{M}}^{\frac{1}{2}} \\
&= \sup_{e \in \mathcal{H}, \, \|e\|=1} \|\varphi e\|_{\mathrm{BMO}_2(\mathbb{T};\mathcal{H})}
\end{aligned}$$

where I runs over all dyadic interval of $\mathbb{T}$ and $\mathrm{BMO}_2(\mathbb{T};\mathcal{H})$ is the usual $\mathcal{H}$-valued dyadic BMO space on $\mathbb{T}$. Thus $\|\cdot\|_{\mathrm{BMO_c}}$ is a norm modulo constant functions. We then define $\mathrm{BMO_c}(\mathbb{T};\mathcal{M})$ as the completion of all φ such that $\|\varphi\|_{\mathrm{BMO_c}} < \infty$. This is a Banach space. $\mathrm{BMO_r}(\mathbb{T};\mathcal{M})$ is defined to be the space of all φ such that $\varphi^* \in \mathrm{BMO_c}(\mathbb{T};\mathcal{M})$ with the norm $\|\varphi\|_{\mathrm{BMO_r}} = \|\varphi^*\|_{\mathrm{BMO_c}}$. Finally, set

$$\mathrm{BMO_{cr}}(\mathbb{T};\mathcal{M}) = \mathrm{BMO_c}(\mathbb{T};\mathcal{M}) \cap \mathrm{BMO_r}(\mathbb{T};\mathcal{M})$$

with the intersection norm

$$\|\varphi\|_{\mathrm{BMO_{cr}}} = \max\{\|\varphi\|_{\mathrm{BMO_c}}, \, \|\varphi\|_{\mathrm{BMO_r}}\}.$$

The following interpolation result is due to Musat [15, Theorem 3.11].

Lemma 2.4 *(Musat) Let $1 < p < \infty$. Then*

$$(\mathrm{BMO}_{cr}(\mathbb{T};\mathcal{M}), \ L^p(\mathbb{T}, L^p(\mathcal{M}))_{p/q} = L^q(\mathbb{T}, L^q(\mathcal{M}))$$

with equivalent norms. Moreover, the relevant equivalence constants depend only on p, q.

The following Burkholder-Gundy inequality is due to Pisier/Xu [23, Theorem 3.11]: Recall that the square function of $\varphi \in L^p(\mathbb{T}, L^p(\mathcal{M}))$ is defined as

$$S(\varphi) = (\sum_k |d_k\varphi|^2)^{\frac{1}{2}}.$$

Lemma 2.5 *(Pisier/Xu) For $1 < p < 2$, we have*

$$\|f\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \simeq^{c_p} \inf_{f=f_1+f_2}\{||S(f_1)||_{L^p(\mathbb{T},L^p(\mathcal{M}))} + ||S(f_2^*)||_{L^p(\mathbb{T},L^p(\mathcal{M}))}\}.$$

For $2 \leq p < \infty$, we have

$$\|f\|_{L^p(\mathbb{T},L^p(\mathcal{M}))} \simeq^{c_p} \max\{||S(f)||_{L^p(\mathbb{T},L^p(\mathcal{M}))}, ||S(f^*)||_{L^p(\mathbb{T},L^p(\mathcal{M}))}\}.$$

The relevant equivalence constants depend only on p.

The space $\mathrm{BMO}_{\mathcal{M}}(\mathbb{T}, \mathcal{M})$

The space $\mathrm{BMO}_{\mathcal{M}}(\mathbb{T}, \mathcal{M})$ appeared in the study of Banach space valued harmonic analysis. Consider an $\mathcal{M}$-valued Bochner integrable function φ, set

$$\varphi_{\mathrm{BMO}_{\mathcal{M}}} = \sup_I (\frac{1}{|I|}\int_I ||\varphi - \varphi_I||^2_{\mathcal{M}} dt)^{\frac{1}{2}}$$

where again I runs over all dyadic interval of $\mathbb{T}$. We then define $\mathrm{BMO}_{\mathcal{M}}(\mathbb{T}; \mathcal{M})$ as the space of all φ such that $\|\varphi\|_{\mathrm{BMO}_{\mathcal{M}}} < \infty$. It is an easy observation that

$$\|\varphi\|_{\mathcal{BMO}_{cr}} \leq \|\varphi\|_{BMO_{\mathcal{M}}} \tag{2.6}$$

$\mathrm{BMO}_{\mathcal{M}}(\mathbb{T}, \mathcal{M})$ is related to the following Hardy space $H^1_{max}(\mathbb{T}, L^1(\mathcal{M}))$,

$$H^1_{max}(\mathbb{T}, L^1(\mathcal{M})) = \{f \in L^1(\mathbb{T}, L^1(\mathcal{M})) \text{ s.}t. \ ||f||_{H^1_{max}} = ||Mf||_{L^1(\mathbb{T})} < \infty\}$$

where Mf is the maximal function of f: $Mf = \sup_n ||E_n f||_{L^1(\mathcal{M})}$. In fact, J. Bourgain (see [2]) and Garcia-Cuerva proved independently that $\mathrm{BMO}_{norm}(\mathbb{T}, \mathcal{M})$ embeds continuously into the dual of the Hardy space $H^1_{max}(\mathbb{T}, L^1(\mathcal{M}))$. That is

$$\tau E\varphi f^* \leq c||\varphi||_{\mathrm{BMO}_{\mathcal{M}}}||f||_{H^1_{max}}.$$

Here E means the integral on $\mathbb{T}$ with respect to dt. We also need the following Doob's inequality for $L^p(\mathcal{M})$-valued function

$$\left\|\sup_{n\in\mathbb{N}} ||E_n f||_{L^p(\mathcal{M})}\right\|_{L^p(\mathbb{T})} \leq \frac{cp}{p-1}\|f\|_{L^p(\mathbb{T},L^p(\mathcal{M}))}, \tag{2.7}$$

for all $1 < p \leq \infty$.

3 Proof of the Main Results

Operator-valued Λ_φ has been studied by Blasco and Pott (see [1]), where Theorem 1.1 was proved for $p = 2$. As in [1], we start by prove the following lemma.

Lemma 3.6 *For $f \in L^p(\mathbb{T}, L^p(\mathcal{M})), g \in L^q(\mathbb{T}, L^q(\mathcal{M})), \frac{1}{p} + \frac{1}{q} = 1$, we have*

$$E \sup_m \left\| \sum_{k=1}^m (d_k f)(d_k g^*) \right\|_{L^1(\mathcal{M})} \leq c_p ||f||_{L^p(\mathbb{T}, L^p(\mathcal{M}))} ||g||_{L^q(\mathbb{T}, L^q(\mathcal{M}))}. \tag{3.8}$$

Proof. Without loss of generality, we assume $q \leq p$. Then $q \leq 2$. Fix a function $g \in L^q(\mathbb{T}, L^q(\mathcal{M}))$. By Lemma 2.5, we can choose g_1, g_2 such that

$$g = g_1 + g_2, \text{ and } ||S(g_1)||_{L^q(\mathbb{T}, L^q(\mathcal{M}))} + ||S(g_2^*)||_{L^q(\mathbb{T}, L^q(\mathcal{M}))} \leq c_q ||g||_{L^q(\mathbb{T}, L^q(\mathcal{M}))} + \varepsilon$$

Therefore, by Lemma 2.3 and Lemma 2.5,

$$\begin{aligned}
& E \sup_m || \sum_{k=1}^m d_k f d_k g^* ||_{L^1(\mathcal{M})} \\
\leq \;& E \sup_m || \sum_{k=1}^m d_k f d_k g_1^* ||_{L^1(\mathcal{M})} + E \sup_m || \sum_{k=1}^m d_k f d_k g_2^* ||_{L^1(\mathcal{M})} \\
\leq \;& E(||S(f)||_{L^p(\mathcal{M})} ||S(g_1)||_{L^q(\mathcal{M})}) + E(||S(f^*)||_{L^p(\mathcal{M})} ||S(g_2^*)||_{L^q(\mathcal{M})}) \\
\leq \;& ||S(f)||_{L^p(\mathbb{T}, L^p(\mathcal{M}))} ||S(g_1)||_{L^q(\mathbb{T}, L^q(\mathcal{M}))} + ||S(f^*)||_{L^p(\mathbb{T}, L^p(\mathcal{M}))} ||S(g_2^*)||_{L^q(\mathbb{T}, L^q(\mathcal{M}))} \\
\leq \;& c_p(c_q + \varepsilon) ||f||_{L^p(\mathbb{T}, L^p(\mathcal{M}))} ||g||_{L^q(\mathbb{T}, L^q(\mathcal{M}))}.
\end{aligned}$$

Let $\varepsilon \to 0$, we prove the lemma.

Proof of Theorem 1.1. Since $(\Lambda_\varphi)^* = \Lambda_{\varphi^*}$ and $||\varphi||_{\mathrm{BMO}_\mathcal{M}} = ||\varphi^*||_{\mathrm{BMO}_\mathcal{M}}$, we only need to prove the Lemma for $p \geq 2$, the other part can be deduced by passing to the adjoint operator. Note that $(d_k \varphi)(d_k f)$ is σ_{k-1} measurable for every $k \in \mathbb{N}$, we have

$$\begin{aligned}
& ||\Lambda_\varphi(f)||_{L^p(\mathbb{T}, L^p(\mathcal{M}))} \\
= \;& || \sum_{k=1}^\infty (d_k \varphi)(E_{k-1} f) + (d_k \varphi)(d_k f) ||_{L^p(\mathbb{T}, L^p(\mathcal{M}))} \\
= \;& \sup_{||g||_{L^q} \leq 1} \tau E \left(\sum_{k=1}^\infty (d_k \varphi)(E_{k-1} f)(d_k g^*) + \sum_{k=1}^\infty (d_k \varphi)(d_k f)(E_{k-1} g^*) \right) \\
= \;& \sup_{||g||_{L^q} \leq 1} \tau E \varphi \left(\sum_{k=1}^\infty (E_{k-1} f)(d_k g^*) + \sum_{k=1}^\infty (d_k f)(E_{k-1} g^*) \right).
\end{aligned}$$

By (2.7), we get

$$||\Lambda_\varphi(f)||_{L^p(\mathbb{T}, L^p(\mathcal{M}))}$$

$$
\begin{aligned}
&\leq \quad ||\varphi||_{BMO_{\mathcal{M}}} \sup_{||g||_{L^q}\leq 1} \left\| \sum_{k=1}^{\infty}(E_{k-1}f)(d_kg^*) + \sum_{k=1}^{\infty}(d_kf)(E_{k-1}g^*) \right\|_{H^1_{\max}} \\
&= \quad ||\varphi||_{BMO_{\mathcal{M}}} \sup_{||g||_{L^q}\leq 1} E\sup_m \left\| \sum_{k=1}^{m}(E_{k-1}f)(d_kg^*) + \sum_{k=1}^{m}(d_kf)(E_{k-1}g^*) \right\|_{L^1(\mathcal{M})} \\
&= \quad ||\varphi||_{BMO_{\mathcal{M}}} \sup_{||g||_{L^q}\leq 1} E\sup_m \left\| (E_mf)(E_mg)^* - \sum_{k=1}^{m}(d_kf)(d_kg^*) \right\|_{L^1(\mathcal{M})}. \qquad (3.9)
\end{aligned}
$$

By the previous lemma and Doob's inequality (2.7), we get

$$
\begin{aligned}
& E\sup_m \left\| (E_mf)(E_mg)^* - \sum_{k=1}^{m}(d_kf)(d_kg^*) \right\|_{L^1(\mathcal{M})} \\
\leq\ & E\sup_m ||(E_mf)(E_mg)^*||_{L^1(\mathcal{M})} + E\sup_m ||\sum_{k=1}^{m} d_kfd_kg^*||_{L^1(\mathcal{M})} \\
\leq\ & E(\sup_m ||E_mf||_{L^p(\mathcal{M})} \sup_m ||E_mg^*||_{L^q(\mathcal{M})}) + c_p||f||_{L^p(\mathbb{T},L^p(\mathcal{M}))}||g||_{L^q(\mathbb{T},L^q(\mathcal{M}))} \\
\leq\ & \left\| \sup_m ||E_mf||_{L^p(\mathcal{M})} \right\|_{L^p(\mathbb{T})} \left\| \sup_m ||E_mg||_{L^q(\mathcal{M})} \right\|_{L^q(\mathbb{T})} + c_p||f||_{L^p(\mathbb{T},L^p(\mathcal{M}))}||g||_{L^q(\mathbb{T},L^q(\mathcal{M}))} \\
\leq\ & c_p||f||_{L^p(\mathbb{T},L^p(\mathcal{M}))}||g||_{L^q(\mathbb{T},L^q(\mathcal{M}))}.
\end{aligned}
$$

Combining (3.9) and the inequality above we prove Theorem1.1.

The following lemma is proved in [13] (Lemma 3.4)

Lemma 3.7

$$
||\pi_\varphi||_{L^\infty(\mathbb{T},\mathcal{M})\to BMO_{cr}(\mathbb{T},\mathcal{M})} \leq c_p(||\pi_\varphi||_{L^p(\mathbb{T},L^p(\mathcal{M}))\to L^p(\mathbb{T},L^p(\mathcal{M}))} + ||\varphi||_{BMO_r(\mathbb{T},\mathcal{M})}).
$$

Proof of Theorem 1.2. Assume $\|\varphi\|_{BMO_{\mathcal{M}}} < \infty$ and for some $1 < p_0 < \infty$,

$$
||\pi_\varphi||_{L^{p_0}(\mathbb{T},L^{p_0}(\mathcal{M}))\to L^{p_0}(\mathbb{T},L^{p_0}(\mathcal{M}))} < \infty.
$$

By Lemma 3.7, we get

$$
\begin{aligned}
||\pi_\varphi||_{L^\infty\to BMO_{cr}} &\leq c_{p_0}(||\pi_\varphi||_{L^{p_0}(\mathbb{T},L^{p_0}(\mathcal{M}))\to L^{p_0}(\mathbb{T},L^{p_0}(\mathcal{M}))} + ||\varphi||_{BMO_r}) \\
&\leq c_{p_0}(||\pi_\varphi||_{L^{p_0}\to L^{p_0}} + ||\varphi||_{BMO_{\mathcal{M}}(\mathbb{T},\mathcal{M})}) < \infty. \qquad (3.10)
\end{aligned}
$$

By Musat's interpolation result Lemma 2.4, we get

$$
||\pi_\varphi||_{L^p(\mathbb{T},L^p(\mathcal{M}))\to L^p(\mathbb{T},L^p(\mathcal{M}))} < \infty, \qquad (3.11)
$$

for any $p_0 < p < \infty$. Note

$$
\Lambda_\varphi = \pi_\varphi + (\pi_{\varphi^*})^*.
$$

By Theorem 1.1 and the identity above, we get

$$\begin{aligned} ||(\pi_{\varphi^*})^*||_{L^p(\mathbb{T},L^p(\mathcal{M}))\to L^p(\mathbb{T},L^p(\mathcal{M}))} &\leq ||\Lambda_\varphi||_{L^p\to L^p} + ||\pi_\varphi||_{L^p\to L^p} \\ &\leq c_p \|\varphi\|_{BMO_{\mathcal{M}}} + ||\pi_\varphi||_{L^p\to L^p} < \infty. \end{aligned} \tag{3.12}$$

for any $p_0 < p < \infty$. Passing to the dual, we have

$$||\pi_{\varphi^*}||_{L^q(\mathbb{T},L^q(\mathcal{M}))\to L^q(\mathbb{T},L^q(\mathcal{M}))} < \infty. \tag{3.13}$$

for all $1 < q < q_0$ with $\frac{1}{q_0} + \frac{1}{p_0} = 1$. Now choose a p_1 with $1 < p_1 < q_0$, repeat all the procedures above with φ, p_0 replaced by φ^*, p_1, we get

$$||\pi_\varphi||_{L^p(\mathbb{T},L^p(\mathcal{M}))\to L^p(\mathbb{T},L^p(\mathcal{M}))} < \infty. \tag{3.14}$$

for all $1 < p < q_1$ with $\frac{1}{q_1} + \frac{1}{p_1} = 1$. Because of the arbitrariness of p_1 we get

$$||\pi_\varphi||_{L^p(\mathbb{T},L^p(\mathcal{M}))\to L^p(\mathbb{T},L^p(\mathcal{M}))} < \infty. \tag{3.15}$$

for all $1 < p < \infty$. This completes the proof. ∎

As mentioned before, when $\mathcal{M} = \mathbb{C}$, the condition $\varphi \in \mathrm{BMO}_{\mathcal{M}}(\mathbb{T}, \mathcal{M})$ in Theorem 1.2 is not necessary since we have

$$||\varphi||_{\mathrm{BMO}_{\mathcal{M}}(\mathbb{T},\mathcal{M})} \leq c||\pi_\varphi||_{L^p\to L^p}, \tag{3.16}$$

for any p. But (3.16) does not hold for general von Neumann algebra $\mathcal{M}$ unless we replace $||\varphi||_{\mathrm{BMO}_{\mathcal{M}}(\mathbb{T},\mathcal{M})}$ by a smaller norm $||\varphi||_{\mathcal{BMO}_c(\mathbb{T},\mathcal{M})}$.

Open Question. Can we remove the assumption $\varphi \in \mathrm{BMO}_{\mathcal{M}}(\mathbb{T}, \mathcal{M})$ in Theorem 1.2?

4 Sharp estimate of the L^2 bounds of Λ_φ.

It is nature to ask if we can replace the $\mathrm{BMO}_{\mathcal{M}}(\mathbb{T}, \mathcal{M})$ norm in Theorem 1.1 by a noncommutative analogue. If yes, we can also do so in Theorem 1.2. Since $\mathrm{BMO}_{\mathcal{M}}(\mathbb{T}, \mathcal{M})$ embeds into the dual of $H^1_{max}(\mathbb{T}, L^1(\mathcal{M}))$ continuously, we consider the dual of the noncommutative Hardy space $H^1_{n.c.m.}(\mathbb{T}, L^1(\mathcal{M}))$ as a noncommutative analogue of $\mathrm{BMO}_{\mathcal{M}}(\mathbb{T}, \mathcal{M})$. $H^1_{n.c.m.}(\mathbb{T}, L^1(\mathcal{M}))$ was studied by Junge/Xu (see [8]) characterized by the noncommutative maximal L^1 norm. The noncommutative maximal norm was introduced by Pisier and Junge. It becomes a central subject in the study of noncommutative martingales mainly due to Pisier, Junge/Xu and their coauthors(see [7],[8], [9],[10], [3], etc.). We recall those definitions in the following. For a sequence $(a_k)_{k=1}^\infty \in L^p(\mathcal{M}), 1 \leq p < \infty$, define

$$||(a_k)_k||_{L^p(\mathcal{M},\ell^\infty)} = \inf\{||A||_{L^p(\mathcal{M})} \mid A \geq \frac{a_k^* + a_k}{2} \geq -A, A \geq i\frac{a_k^* - a_k}{2} \geq -A, \forall k\}.$$

Set

$$H^p_{n.c.m.}(\mathbb{T}, L^p(\mathcal{M})) = \{f \in L^p(\mathbb{T}, L^p(\mathcal{M})), ||f||_{H^p_{n.c.m.}} = ||((E_nf))_n||_{L^p(L^\infty(\mathbb{T})\otimes\mathcal{M},\ell^\infty)} < \infty\}.$$

Note the definition of the norm $L^p(\mathcal{M}, \ell^\infty)$ given in (4.17) is different from but equivalent to the original definition given in [20], [7]. And for $a_k \geq 0$,

$$||(a_k)_k||_{L^p(\mathcal{M},\ell^\infty)} = \inf\{||A||_{L^p(\mathcal{M})} \mid A \geq a_k, \forall k\}. \tag{4.17}$$

A noncommutative Doob's inequality was proved by Junge (see [7]). In particular, for any $L^p(\mathcal{M})$ valued function f defined on $\mathbb{T}$, we have

Lemma 4.8 *(M. Junge)*

$$||(E_nf)_n||_{L^p(L^\infty(\mathbb{T})\otimes\mathcal{M},\ell^\infty)} \leq \frac{c}{(p-1)^2}||f||_{L^p(\mathbb{T},L^p(\mathcal{M}))}.$$

Note, the power "2" on $p-1$ is not removable in the inequality above.

In the following, we show that the answer to the question asked at the beginning of the section is negative. We can not dominate the $L^p(\mathbb{T}, L^p(\mathcal{M}))$ bounds of Λ_φ by $||\varphi||_{(H^1_{n.c.m.})^*}$. Here

$$||\varphi||_{(H^1_{n.c.m.})^*} = \sup\{\tau\int\varphi^* f dt; ||f||_{H^1_{n.c.m.}} \leq 1\}.$$

From now on, our von Neumann algebra $\mathcal{M}$ will be M_N, the algebra of all N by N matrices with the usual trace tr. And $L^p(\mathcal{M})$'s become S^p_N's the Schatten p classes on ℓ^2_N. We have the following sharp estimate of $||\Lambda_\varphi||_{L^2(\mathbb{T},S^2_N)\to L^2(\mathbb{T},S^2_N)}$ by the $||\cdot||_{(H^1_{n.c.m.})^*}$ norm according to N.

Theorem 4.9 *For an M_N-valued function φ, we have*

$$\|\Lambda_\varphi\|_{L^2(\mathbb{T},S^2_N)\to L^2(\mathbb{T},S^2_N)} \leq c(\log N)^2 \, \|\varphi\|_{(H^1_{n.c.m.})^*}.$$

And the constant $c(\log N)^2$ is sharp.

Lemma 4.10 *For any $f \in L^2(\mathbb{T}, S^2_N)$,*

$$||(|E_nf|^2)_n||_{L^1(L^\infty(\mathbb{T})\otimes M_N,\ell^\infty)} \leq c(N)||f||_{L^2(\mathbb{T},S^2_N)}. \tag{4.18}$$

with $c(N) = c(\log N)^2$ and the constant is sharp.

Proof of Lemma 4.10. Without loss of generality, assume $||f||_{L^2(\mathbb{T},S^2_N)} = 1$. Fix a pair of conjugate indices $p, q, p < 2; \frac{1}{p} + \frac{1}{q} = 1$. We decompose $|E_nf|^2$ as follows:

$$|E_nf|^2 = |E_nf|^{\frac{1}{p}}|E_nf|^{\frac{2}{q}}|E_nf|^{\frac{1}{p}} \leq |E_nf|^{\frac{1}{p}}||E_nf||^{\frac{2}{q}}_{M_N}|E_nf|^{\frac{1}{p}}.$$

Note we always have $||\cdot||_{M_N} \leq ||\cdot||_{S_N^2}$ and $||E_nf||_{S_N^2} \leq E_n||f||_{S_N^2}$ because of the convexity of the norm $||\cdot||_{S_N^2}$. We get

$$|E_nf|^2 \leq |E_nf|^{\frac{1}{p}}||E_nf||_{S_N^2}^{\frac{2}{q}}|E_nf|^{\frac{1}{p}} \leq (E_n||f||_{S_N^2})^{\frac{2}{q}}|E_nf|^{\frac{2}{p}}.$$

By the convexity of the operator valued function $x \to |x|^s$ for $1 < s \leq 2$, we also have $|E_nf|^{\frac{2}{p}} \leq E_n|f|^{\frac{2}{p}}$. Thus, we get

$$|E_nf|^2 \leq (E_n||f||_{S_N^2})^{\frac{2}{q}}E_n|f|^{\frac{2}{p}}.$$

Let

$$g = |f|^{\frac{2}{p}}.$$

Then $(E_ng)_n$ is an matrix valued martingale with L^p norm as 1. Note $E_ng \geq 0$, by Lemma 4.8 and the interpretation (4.17), there exits a G such that $G \geq E_ng$ and

$$||G||_{L^p(\mathbb{T},S_N^p)} \leq \frac{c}{(p-1)^2}.$$

On the other hand, apply the classical Doob's inequality to $(E_n||f||_{S_N^2})_n$, we have

$$\left\|\sup_n E_n||f||_{S_N^2}\right\|_{L^2(\mathbb{T})} \leq c$$

with an absolute constant c. Let $H = (\sup_n E_n||f||_{S_N^2})^{\frac{2}{q}}$, we have $||H||_{L^q} \leq c$ and

$$||H \otimes I_N||_{L^q(\mathbb{T},S_N^q)} \leq c||I_N||_{S_N^q} \leq cN^{\frac{1}{q}}.$$

Set $F = (H \otimes I_N)G$, we get $|E_nf|^2 \leq F$ and

$$||F||_{L^1(\mathbb{T},S_N^1)} \leq ||H \otimes I_N||_{L^q(\mathbb{T},S_N^q)}||G||_{L^p(\mathbb{T},S_N^p)} \leq \frac{cN^{\frac{1}{q}}}{(p-1)^2}.$$

Now choose $q = 2 + 2\ln N$, we get

$$||F||_{L^1(S_N^1)} \leq cN^{\frac{1}{2\ln N+2}}(\ln N)^2 \leq c(\ln N)^2.$$

Therefore,

$$||(|E_nf|^2)_n||_{L^1(L^\infty(\mathbb{T})\otimes M_N,\ell^\infty)} \leq c(\ln N)^2.$$

To prove the sharpness, choose a sequence $(\alpha_k)_{k=1}^N$ in the unit ball of ℓ_N^2. Let

$$d_kf = e_{1,k} \otimes \alpha_k r_k$$

with r_k the kth Rademacher function on $\mathbb{T}$. Then we find

$$||f||_{L^2(\mathbb{T},S_N^2)} = ||\alpha||_{\ell_N^2} = 1, \ |E_nf|^2 = P_n(\alpha \otimes \alpha)$$

where P_n is the projection on the first n columns and n rows. By (4.18) we get

$$||P_n(\alpha \otimes \alpha)||_{L^1(M_N, \ell^\infty)} \leq c(N)$$

for any $\alpha = (\alpha_k)_{k=1}^N$ in the unit ball of ℓ_N^2. Note the unit ball of S_N^1 is the in the convex hall of the set of all these $\alpha \otimes \alpha$. We deduce that

$$||P_n(A)||_{L^1(M_N, \ell^\infty)} \leq c(N)||A||_{S_N^1}, \tag{4.19}$$

for all $A \in S_N^1$. We need to show that the constant $c(N)$ such that (4.19) holds is bigger than $c(\ln N)^2$. This is known to experts of noncommutative maximal norm. For completion, we give a proof of this estimation following an idea used in [8]. We consider the Hilbert matrix $h = (h_{i,j})_{1 \leq i,j \leq N} \in M_N$ defined by

$$h_{i,j} = \begin{cases} (j-i)^{-1} & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases}.$$

It is well known that (see [12])

$$||h||_{M_N} \leq c \text{ and } ||Th||_{M_N} \approx \ln(N+1), \tag{4.20}$$

where T is the triangle projection. Now let h_k be the matrix whose kth row is that of h and all others are zero. Set

$$g_k = h_k^* h_k.$$

Thus

$$\sum_{k=1}^N g_k = \sum_{k=1}^N h_k^* h_k = (h_1^*, h_2^*, \cdots h_N^*) \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix}$$

and

$$||\sum_{k=1}^N g_k||_{M_N} = ||(h_1^*, h_2^*, \cdots h_N^*)||_{M_{N^2}}^2 = ||h||_{M_N}^2. \tag{4.21}$$

On the other hand,

$$\sum_{k=1}^N P_k g_k = \sum_{k=1}^N P_k(h_k^* h_k) = \sum_{k=1}^N (P_k h_k^*)(P_k h_k) = (P_1 h_1^*, P_2 h_2^*, \cdots P_N h_N^*) \begin{pmatrix} P_1 h_1 \\ P_2 h_2 \\ \vdots \\ P_N h_N \end{pmatrix}.$$

and

$$||\sum_{k=1}^N P_k g_k||_{M_N} = ||(P_1 h_1^*, P_2 h_2^*, \cdots P_N h_N^*)||_{M_{N^2}}^2 = ||Th||_{M_N}^2.$$

Therefore

$$\begin{aligned}
||Th||_{M_N}^2 &= ||\sum_{k=1}^N P_k g_k||_{M_N} \\
&= \sup_{A\geq 0, ||A||_{S_N^1}\leq 1} tr \sum_{k=1}^N (P_k g_k) A \\
&= \sup_{A\geq 0, ||A||_{S_N^1}\leq 1} tr \sum_{k=1}^N g_k (P_k A) \\
&\leq \sup_{A\geq 0, ||A||_{S_N^1}\leq 1} \inf_{\widetilde{A}\geq P_k A} tr \sum_{k=1}^N g_k \widetilde{A} \\
&\leq ||\sum_{k=1}^N g_k||_{M_N} \sup_{A\geq 0, ||A||_{S_N^1}\leq 1} \inf_{\widetilde{A}\geq P_k A} ||\widetilde{A}||_{S_N^1}
\end{aligned}$$

By the interpretation (4.17) and (4.21) we get

$$\begin{aligned}
||Th||_{M_N}^2 &\leq ||\sum_{k=1}^N g_k||_{M_N} \sup_{||A||_{S_N^1}\leq 1} ||(P_n A)_n||_{L^1(M_N, \ell^\infty)} \\
&\leq ||h||_{M_N}^2 c(N). \qquad (4.22)
\end{aligned}$$

Combining (4.20) and (4.22), we then get

$$c(N) \geq c(\log N)^2.$$

This finishes the proof.

Proof of Theorem 4.9. As in the proof of Theorem 1.1, passing by duality we see $\|\Lambda_\varphi\|_{L^2(\mathbb{T},S_N^2)\to L^2(\mathbb{T},S_N^2)} \leq c(N)\,\|\varphi\|_{(H^1_{n.c.m.})^*}$ if and only if

$$||(E_n f E_n g^* - \sum_{k=1}^n d_k f d_k g^*)_n||_{L^1(S_N^1, \ell_\infty)} \leq c(N) ||f||_{L^2(\mathbb{T},S_N^2)} ||g||_{L^2(\mathbb{T},S_N^2)} \qquad (4.23)$$

for any $f, g \in L^2(\mathbb{T}, S_N^2)$. Note

$$||(\sum_{k=1}^n d_k f d_k f^*)_n||_{L^1(L^\infty(\mathbb{T})\otimes M_N, \ell_\infty)} = ||S^2(f^*)||_{L^1(\mathbb{T},S_N^1)} = ||f||^2_{L^2(\mathbb{T},S_N^2)}.$$

By polarization, we get

$$||(\sum_{k=1}^n d_k f d_k g^*)_n||_{L^1(L^\infty(\mathbb{T})\otimes M_N, \ell_\infty)} \leq ||f||_{L^2(\mathbb{T},S_N^2)} ||g||_{L^2(\mathbb{T},S_N^2)}.$$

Therefore the condition (4.23) is equivalent to

$$||(E_n f E_n g^*)_n||_{L^1(L^\infty(\mathbb{T})\otimes M_N,\ell_\infty)} \leq c(N)||f||_2||g||_2,$$

for any $f, g \in L^2(\mathbb{T}, S_N^2)$. By polarization again, this is equivalent to

$$||(E_n f E_n f^*)_n||_{L^1(L^\infty(\mathbb{T})\otimes M_N,\ell_\infty)} \leq c(N)||f||^2_{L^2(\mathbb{T},S_N^2)}.$$

for any $f \in L^2(\mathbb{T}, S_N^2)$. The theorem is followed by the previous lemma.

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